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Spin-glass order parameter on a Bethe lattice

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Abstract. An order parameter function for the spin-glass state on a Bethe lattice of coordination number z is calculated by the use of hierarchical boundary conditions. The order parameter is discrete for finite z and becomes a continuous function in the limit $z \to \infty$. Numerical calculations are performed at zero temperature and zero external field for several values of z.

The most widely studied model for a spin glass is the infinite-ranged Sherrington-Kirkpatrick (SK) model [1]. It is a mean-field model in which each spin interacts with every other spin. The solution of the model is obtained by means of the replica trick [2] and it may be symmetric or non-symmetric with respect to the replicas. The replica symmetric solution [1] is shown to be unstable below the de Almeida–Thouless (AT) instability line [3]. Below this line, a replica-symmetry breaking sets in and the system displays a spin-glass state described by Parisi's order parameter function [4], a generalisation of the Edwards–Anderson order parameter [2].

An alternative mean-field approach to the SK model is the study of spin glasses on a Bethe lattice [5–7]. By a Bethe lattice we mean the local properties of an infinite Cayley tree. One important feature of the spin glass defined on such a lattice is that the probability distribution of the total effective field is non-Gaussian, as long as the coordination number z is finite. In the limit $z \to \infty$, however, this distribution turns out to be Gaussian [8] and we expect, in this case, to obtain results similar to those of the SK model. In fact, it has been shown [8] that in this limit the solution is identical to the SK replica-symmetric solution. To obtain the instability line one should analyse the correlation between two replicas of the Bethe lattice as was shown by Thouless [9]. It is also possible to prove explicitly [10] that in the limit $z \to \infty$ the instability line is identical to the AT line.

To set up an order parameter function for the spin-glass state in the Bethe lattice it is necessary to study the correlation between the two replicas with appropriate boundary conditions [11]. In the limit $z \to \infty$ Fyodorov [11] has obtained an order parameter function $S(x)$ which is continuous and monotonic in the interval $0 \leq x \leq 1$. In this paper we obtain an order parameter function on a Bethe lattice for the case of finite coordination number. We follow Fyodorov's approach [11] to show that the order parameter will be a function of a discrete variable l that takes the values 0, 1, 2, ..., $z-1$. We analyse here only the case of zero external field and zero temperature.

Consider a Bethe lattice of coordination number $z = r+1$. Each bond can take the values $+J$ or $-J$ with equal probability. For the case of zero external field, considered here, it is possible to flip some spins in an appropriate manner so that all bonds become ferromagnetic. We suppose, therefore, that the spins interact ferromagnetically. Let us denote by h_{ij} the effective field over the site i due to the site j of the next generation. The magnetisation m_0 of the spin in the centre of the Cayley tree will be given by

$$m_0 = \tanh \beta \sum_{i=1}^{z} h_{0i} \tag{1}$$

and the effective fields are obtained recursively through the equation [6]

$$h_{ij} = f\left(\sum_{k=1}^{r} h_{jk}\right) \tag{2}$$

where

$$f(H) = \beta^{-1} \tanh^{-1}(\tanh \beta J \tanh \beta H). \tag{3}$$

By treating h_{ij} as random variables it follows that the probability distribution $g_{ij}(h_{ij})$ of the effective field h_{ij} satisfies the following integral equation

$$g(h_{ij}) = \int \delta\left(h_{ij} - f\left(\sum_{k=1}^{r} h_{jk}\right)\right) \prod_{k=1}^{r} g_{jk}(h_{jk})\, \mathrm{d}h_{jk}. \tag{4}$$

Following Thouless [9], we consider now two replicas A and B of the Bethe lattice. If we denote by $G_{ij}(h_{ij}^{\mathrm{A}}, h_{ij}^{\mathrm{B}})$ the joint probability distribution function of the effective fields h_{ij}^{A} and h_{ij}^{B}, associated with replicas A and B, respectively, then the following equation can be obtained:

$$G_{ij}(h_{ij}^{\mathrm{A}}, h_{ij}^{\mathrm{B}}) = \int \delta\left(h_{ij}^{\mathrm{A}} - f\left(\sum_{k=1}^{r} h_{jk}^{\mathrm{A}}\right)\right) \delta\left(h_{ij}^{\mathrm{B}} - f\left(\sum_{k=1}^{r} h_{jk}^{\mathrm{B}}\right)\right) \prod_{k=1}^{r} G_{jk}(h_{jk}^{\mathrm{A}}, h_{jk}^{\mathrm{B}})\, \mathrm{d}h_{jk}^{\mathrm{A}}\, \mathrm{d}h_{jk}^{\mathrm{B}}. \tag{5}$$

Notice that, from the property

$$\int G_{ij}(h_{ij}^{\mathrm{A}}, h_{ij}^{\mathrm{B}})\, \mathrm{d}h_{ij}^{\mathrm{B}} = g_{ij}(h_{ij}^{\mathrm{A}}) \tag{6}$$

equation (4) follows from equation (5).

Let us focus on equations (4) and (5) which we write in the short notation

$$g_{ij} = \mathscr{F}_1(g_{j1}, g_{j2}, g_{j3}, \ldots, g_{jr}) \tag{7}$$

and

$$G_{ij} = \mathscr{F}_2(G_{j1}, G_{j2}, G_{j3}, \ldots, G_{jr}). \tag{8}$$

These are recursive equations that should be solved for a given boundary condition. It is clear that by solving equation (8) we are also solving equation (7) due to the property (6). According to Fyodorov [11], it is possible to generate, in a hierarchic way, z sets of boundary conditions which are labelled by $l = 0, 1, 2, \ldots, r = z-1$. For each set, a joint probability distribution $G_l(h_{\mathrm{A}}, h_{\mathrm{B}})$ is obtained, in the limit of an infinite Cayley tree, which satisfies the equation

$$G_l = \mathscr{F}_2(\underbrace{G_r, G_r, \ldots, G_r}_{l}, \underbrace{G_l, G_l, \ldots, G_l}_{r-l}). \tag{9}$$

All distributions $G_l(h_{\mathrm{A}}, h_{\mathrm{B}})$, $l = 0, 1, 2, \ldots, r$, are such that

$$\int G_l(h_{\mathrm{A}}, h_{\mathrm{B}})\, \mathrm{d}h_B = g(h_{\mathrm{A}}) \tag{10}$$

where $g(h)$ is independent of l and satisfies the equation

$$g = \mathscr{F}_1(g, g, \ldots, g). \tag{11}$$

Let us suppose that we have first solved equation (11) so that $g(h)$ is obtained. Then, the distribution $G_r(h_A, h_B)$ is given by

$$G_r(h_A, h_B) = \delta(h_A - h_B)g(h_A) \tag{12}$$

which, of course, solves equation (9) for $l = r$. To get $G_l(h_A, h_B)$ for $l < r$ we use equation (9) with $G_r(h_A, h_B)$ given by equation (12), except for the case $l = 0$ when this function is not necessary. In this case, however, the solution is given by

$$G_0(h_A, h_B) = g(h_A)g(h_B) \tag{13}$$

which is valid only for the case of zero external field.

The order parameter S_l is defined as the correlation $\langle m_0^A m_0^B \rangle$ between the magnetisations m_0^A and m_0^B of replicas A and B, respectively. That is,

$$S_l = \int (\tanh \beta H_A)(\tanh \beta H_B) P_l(H_A, H_B)\, dH_A\, dH_B \tag{14}$$

where

$$P_l(H_A, H_B) = \int \delta\left(H_A - \sum_{i=1}^{z} h_i^A\right) \delta\left(H_B - \sum_{i=1}^{z} h_i^B\right) \times \prod_{i=1}^{l} G_r(h_i^A, h_i^B)\, dh_i^A\, dh_i^B \prod_{i=l+1}^{z} G_l(h_i^A, h_i^B)\, dh_i^A\, dh_i^B. \tag{15}$$

The Edwards-Anderson order parameter q is defined by

$$q = \int (\tanh \beta H)^2 p(H)\, dH \tag{16}$$

where

$$p(H) = \int \delta\left(H - \sum_{i=1}^{z} h_i\right) \prod_{i=1}^{z} g(h_i)\, dhi. \tag{17}$$

From (10) it follows that

$$p(H) = \int P_l(H, H')\, dH'. \tag{18}$$

From these definitions it follows that $S_r = q$ and $S_0 = 0$.

Let us now consider the zero-temperature limit. From now on we set $J = 1$. When $\beta \to \infty$ we have

$$f(H) = \begin{cases} -1 & H \leqslant -1 \\ H & -1 < H < 1 \\ 1 & 1 \leqslant H. \end{cases} \tag{19}$$

We look for solutions $G_l(h, h')$ and $g(h)$ of the types

$$G_l(h, h') = \sum_{\tau=-1}^{1} \sum_{\tau'=-1}^{1} A_l(\tau, \tau')\delta(h - \tau)\delta(h' - \tau') \tag{20}$$

with

$$A_l(0, 0) = a_l \tag{21a}$$

$$A_l(1, 1) = A_l(-1, -1) = b_l/2 \tag{21b}$$

$$A_l(1, -1) = A_l(-1, 1) = c_l/2 \tag{21c}$$

$$A_l(0, 1) = A_l(0, -1) = A_l(1, 0) = A_l(-1, 0) = d_l/4 \tag{21d}$$

and

$$g(h) = a\delta(h) + (b/2)(\delta(h+1) + \delta(h-1)). \tag{22}$$

The coefficients a_l, b_l, c_l, d_l, a, b are to be determined under the normalisation restriction $a_l + b_l + c_l + d_l = 1$ and $a + b = 1$. Due to the property (10) we should impose also that $a_l + d_l/2 = a$ and $b_l + c_l + d_l/2 = b$.

We solve first equation (11) to determine the coefficients a and b. From equation (12) we obtain $a_r = a$, $b_r = b$, $c_r = 0$ and $d_r = 0$. From equation (13) we get $a_0 = a^2$, $b_0 = c_0 = b^2/2$ and $d_0 = 2ab$. For $0 < l < r$, the coefficients were determined by solving equation (9) with the ansatz (20). Numerical values of the coefficients are shown in table 1 for several values of the ramification up to $r = 10$. Table 1 also shows the numerical values of S_l which is obtained from

$$S_l = \int \mathrm{sgn}(H_A)\,\mathrm{sgn}(H_B) P_l(H_A, H_B)\, \mathrm{d}H_A\, \mathrm{d}H_B. \tag{23}$$

Table 1. Values of the weights a_l, b_l, c_l, d_l and the order parameter function S_l for $r = 2$, 3, 4, 5 and 10.

r	l	a_l	b_l	c_l	d_l	S_l
2	0	0.1111	0.2222	0.2222	0.4444	0.0000
	1	0.3333	0.6667	0.0000	0.0000	0.7407
	2	0.3333	0.6667	0.0000	0.0000	0.7407
3	0	0.0400	0.3200	0.3200	0.3200	0.0000
	1	0.1635	0.7606	0.0028	0.0730	0.7162
	2	0.2000	0.8000	0.0000	0.0000	0.7680
	3	0.2000	0.8000	0.0000	0.0000	0.7680
4	0	0.0523	0.2974	0.2974	0.3529	0.0000
	1	0.0898	0.5648	0.0674	0.2781	0.4975
	2	0.2288	0.7712	0.0000	0.0000	0.8120
	3	0.2288	0.7712	0.0000	0.0000	0.8120
	4	0.2288	0.7712	0.0000	0.0000	0.8120
5	0	0.0279	0.3469	0.3469	0.2783	0.0000
	1	0.0364	0.5451	0.1571	0.2615	0.3662
	2	0.0900	0.7327	0.0231	0.1542	0.6771
	3	0.1671	0.8329	0.0000	0.0000	0.8160
	4	0.1671	0.8329	0.0000	0.0000	0.8160
	5	0.1671	0.8329	0.0000	0.0000	0.8160
10	0	0.0188	0.3721	0.3721	0.2369	0.0000
	1	0.0198	0.4696	0.2757	0.2349	0.1900
	2	0.0224	0.5456	0.2023	0.2298	0.3369
	3	0.0273	0.6134	0.1393	0.2200	0.4662
	4	0.0373	0.6798	0.0829	0.1999	0.5894
	5	0.0597	0.7514	0.0338	0.1551	0.7142
	6	0.1107	0.8338	0.0024	0.0531	0.8393
	7	0.1373	0.8627	0.0000	0.0000	0.8763
	8	0.1373	0.8627	0.0000	0.0000	0.8763
	9	0.1373	0.8627	0.0000	0.0000	0.8763
	10	0.1373	0.8627	0.0000	0.0000	0.8763

Figure 1 shows the order parameter S_l as a function of l for $r=5$, 10, 20 and ∞. S_l is a monotonic function of l and has a plateau which vanishes in the limit $r \to \infty$. The result in the limit $r \to \infty$ is obtained as follows.

When $r \gg 1$, the probability distribution $P_l(H_A, H_B)$ approaches the Gaussian distribution [10, 11]

$$P_l(H_A, H_B) = [4\pi^2 r^2 (q^2 - s^2)]^{-1/2} \exp\left(-\frac{(qH_A^2 + qH_B^2 - 2sH_A H_B)}{2r(q^2 - s^2)}\right) \tag{24}$$

with

$$s = \frac{l}{r} q + \left(1 - \frac{l}{r}\right) S_l. \tag{25}$$

From (18) we should have also

$$p(H) = (2\pi r q)^{-1/2} \exp(-H^2/2rq). \tag{26}$$

If we take the limit $\beta \to \infty$, we see that $q \to 1$ and

$$S_l = \int \operatorname{sgn}(\xi_A) \operatorname{sgn}(\xi_B) [4\pi^2 (1 - s^2)]^{-1/2} \exp\left(-\frac{(\xi_A^2 + \xi_B^2 - 2s\xi_A \xi_B)}{2(1 - s^2)}\right) \mathrm{d}\xi_A \, \mathrm{d}\xi_B. \tag{27}$$

After performing this integral we get

$$S_l = \frac{2}{\pi} \sin^{-1}(s). \tag{28}$$

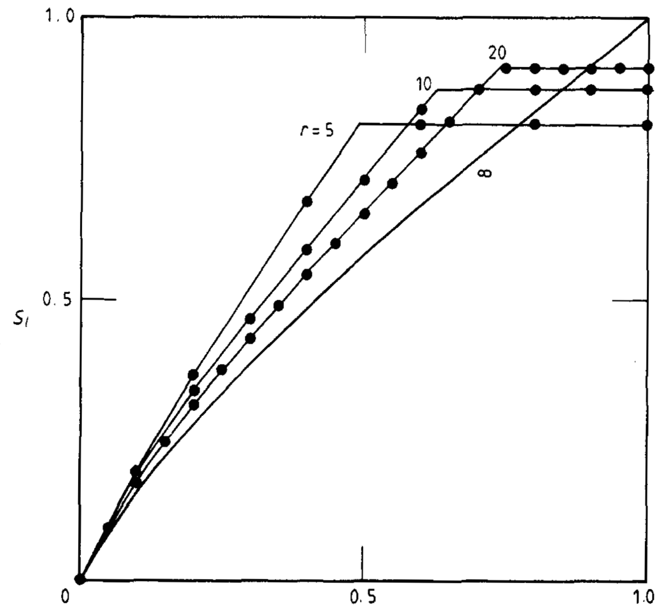

Figure 1. Order parameter function S_l as a function of l/r for $r=5$, 10, 20 and ∞. The line segments joining the dots are merely guides to the eye.

Equations (28) and (25) determine S_l. In the limit $r \to \infty$, the order parameter will be a continuous function of the variable $x = l/r$, defined in the interval [0, 1], and will satisfy the equation

$$S(x) = \frac{2}{\pi} \sin^{-1}[x + (1-x)S(x)]. \tag{29}$$

The solution of this equation is an increasing function of x, shown in figure 1, with $S(0) = 0$, $S(1) = 0$, $S'(0) = 2/(\pi - 2) \simeq 1.7519$ and $S'(1) = 8/\pi^2 \simeq 0.8106$. $S(x)$ may be inverted to yield

$$x = \frac{\sin(\pi S/2) - S}{1 - S}. \tag{30}$$

We point out finally that the type of solution used here may not be unique. There is also a solution for $g(h)$ with a continuous part besides the delta function [6, 12, 13]. However, the use of such a solution will not change the qualitative results we have obtained. In particular, the order parameter S_l will be a discrete function for finite z and will become a continuous function in the limit $z \to \infty$.

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